PERGAMON

International Journal of Solids and Structures 40 (2003) 2851–2873

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Work-conjugacy between rotation-dependent moments and finite rotations

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Received 19 April 2002; received in revised form 13 January 2003

Abstract

In this paper we investigate the work-conjugacy between rotation-dependent moments and finite rotation measures. The methodology adopted consists of writing all the relevant quantities in terms of the rotation vector, using expressions that remain exact in the finite rotation range. Through this procedure, we show how (i) to identify work-conjugate (rotation-dependent) moments and rotation measures, (ii) to derive a necessary condition for moment conservativeness and (iii) to obtain the general form of an isotropic conservative rotation-dependent moment. Several moment and finite rotation definitions that have been used in the past are investigated and, in particular, it is shown that the various existing definitions for the so-called semi-tangential moments are distinct in the finite rotation range and that not all of them are conservative. The tangent operator symmetry is discussed in the context of finite element analysis of conservative systems with rotational degrees of freedom, adopting either an additive or a multiplicative update.

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Keywords: Rotation-dependent moments; Finite rotations; Conservative moments; Semi-tangential rotations

1. Introduction

1.1. Conservative moments and commutative rotations

It is well known that a constant applied moment, the so-called *axial moment*, is not conservative (Ziegler, 1968). However, this does not mean that conservative applied moments do not exist. Instead, it simply means that a conservative moment must be *rotation-dependent*.

It is also well known that rotations about fixed axes are non-commutative. As a consequence, spatial finite elements having such rotations as degrees of freedom lead to non-symmetrical tangent stiffness matrices, even for conservative loadings (Argyris et al., 1978). However, it is possible to describe finite

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doi:10.1016/S0020-7683(03)00078-7

rotations in such a way that the composition of successive rotations is *additive* or, at least, exhibits some kind of *commutativity*.

Although Ziegler (1968) and Argyris and co-workers (Argyris et al., 1978, 1979a; Argyris, 1982) have done much to clarify these issues, most of their work took place in the context of second-order approximations (i.e. including up to linear terms in the moment rotation-dependency) and, thus, they did not explore the finite rotation range thoroughly. The aim of the present paper is to investigate and characterize, in the finite rotation range, several (i) rotation-dependent moments and (ii) rotation descriptions considered in the past.

1.2. Ziegler's definitions of conservative moments

Ziegler (1968) conceived three types of conservative moments, namely *quasi-tangential*, *semi-tangential* and *pseudo-tangential* moments. All these moments can be generated by simple mechanisms involving conservative forces applied either through strings wrapped around disks or directly at the end points of rigid levers (see Fig. 1).

Ziegler's semi-tangential moment is generated when any number $n > 2$ of forces of equal magnitude are uniformly distributed around a disk positioned perpendicularly to the initial moment axis (Ziegler, 1968). The semi-tangential concept is most appealing because the knowledge of the force directions is not required, which means that the semi-tangential moment is completely defined by its initial vector (obviously, the current rotation must also be known) (Argyris et al., 1978, 1979a).

On the other hand, Ziegler's quasi-tangential moment is generated by only two forces (Ziegler, 1968), but its complete definition requires *two* directions (e.g. the disk axis and generating force directions).

In his book, Ziegler (1968) did not explore the implications of these conservative moment definitions in the large rotation range. We remark that his disk and string based definitions are not easily generalizable to large rotations when the disk axis leans towards the strings direction. Moreover, it is not clear whether a holonomic kinematical condition can be established between the disk and the strings. We point out that, if the kinematical condition is non-holonomic, the resulting moment can be non-conservative, even though the forces themselves remain conservative (Lanczos, 1970).

Ziegler (1968) has also introduced the more convenient *pseudo-tangential* conservative moment, stemming from forces applied at the ends of a cross-bar (lever), instead of being tangent to a disk. Although Ziegler considered only one cross-bar to define the pseudo-tangential moment, the concept can be extended to two cross-bars, leading to a moment designated here as *cross semi-tangential moment* (see Fig. 1). Like

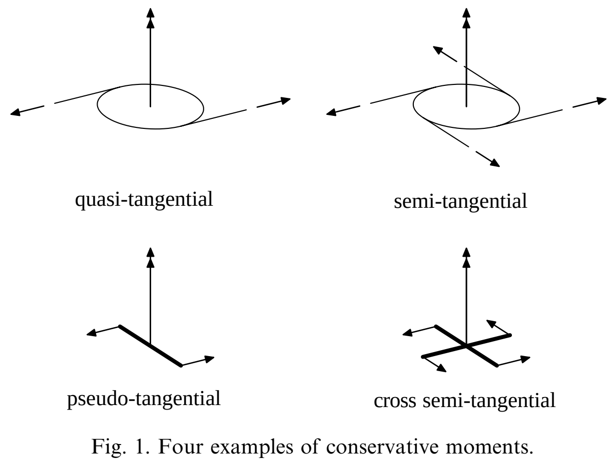

Fig. 1. Four examples of conservative moments.

the (disk based) semi-tangential moment, it depends only on its initial vector, i.e. it is insensitive to the initial lever orientation. Although the two definitions are closely related, they differ in the second-order terms. For instance, when rotations parallel to the initial moment axis are considered, (i) Ziegler's semi-tangential moment remains always unaltered, while (ii) the cross semi-tangential moment becomes null for a 90° rotation. Since Ziegler's semi-tangential moment definition is less sound in the finite rotation range, both from mathematical and physical points of view, it will not be further considered in this work.

1.3. Argyris semi-tangential definitions

Subsequently, Argyris et al. (1978) preferred to use levers, instead of Ziegler's disk, to redefine the semi-tangential moments, which means that they effectively dealt with the above defined cross-tangential moment.

In order to avoid obtaining unsymmetrical tangent matrices in conservative problems, Argyris and co-workers introduced a new rotation definition—the so-called semi-tangential rotations (Argyris et al., 1978, 1979a). Such rotations were, supposedly, work-conjugate to the (cross) semi-tangential moment. Unfortunately, since most of the reasoning was made in the context of small rotations, they presented several semi-tangential moment and rotation definitions which turn out to be clearly distinct in the finite rotation range.

In the course of this paper, we will distinguish between *four* different kinds of moments and rotations, all corresponding to Argyris semi-tangential concept, namely:

(i) the *cross semi-tangential* moment, corresponding to the lever mechanism;
(ii) the *mean semi-tangential* moment, which is the mean value of the axial and follower moments;
(iii) the *commutative semi-tangential* rotation, based on a commutative rule for the rotations;
(iv) the *half semi-tangential* rotation, based on the decomposition of the rotation into two successive equal half-rotations.

For each moment definition, there is a work-conjugate rotation definition and vice versa. So, Argyris semi-tangential concept comprises, in reality, four different definitions.

1.4. The internal moment nature controversy

In the context of the (variationally based) finite element method, the out-of-balance load vector is work-conjugate to the degrees of freedom. This means that, when semi-tangential rotations are used as rotational degrees of freedom, the corresponding equilibrium equations involve semi-tangential moments (Argyris et al., 1978). Since the load vector is often expressed as the difference between internal and external loads, the issue of which is the internal moment nature was raised and has became the source of a lot of controversy.

Argyris et al. (1979b) and Yang and McGuire (1986a,b) identified bending moments and Saint Venant torques as being quasi-tangential and semi-tangential, respectively. These authors also recognized that, adopting such definitions for the internal moment behavior, an apparent loss of equilibrium occurs when an (initially equilibrated) joint is subjected to a finite rotation (Argyris et al., 1978, 1979b; Yang and McGuire, 1986a,b). This inconsistency led them to assume a semi-tangential behavior for all internal moments, in spite of the fact that bending moments seemed to be quasi-tangential. Yang and Kuo (1994), on the other hand, found that the internal moment semi-tangential assumption was unnecessary to derive the tangent stiffness matrix.

Another approach was adopted by Teh and Clarke (1997), who argued that internal moments should be viewed as follower moments (or 'moments of the fourth kind', which is just a second-order approximation of follower moments). This assertion led them to claim later that 'the proper consideration of the rotational behavior of nodal moments invariably leads to asymmetric tangent stiffness matrices for spatial beams' (Teh and Clarke, 1999).

Saleeb et al. (1992, p. 489) began to shed light on this subject by first recognizing that the semi-tangential internal moment behavior was a direct consequence of the kinematic description, and not an intrinsic property. Indeed, as recently pointed out by Izzuddin (2001), (i) internal moments perform work over curvature and twist strains, (ii) element end moments perform work over nodal rotations, (iii) any element end moment definition can be adopted without compromising accuracy and (iv) the adopted nodal moment definition should simply be the one which is work-conjugate to the chosen nodal rotation definition. We remark that several other authors (Simo and Vu-Quoc, 1986; Cardona and Geradin, 1988; Ibrahimbegović et al., 1995; Ritto-Corrêa and Camotim, 2002), who never made an issue out of the internal moment nature, have used precisely work-conjugate nodal end moment and rotation definitions (in the formulation of geometrically exact beam models).

1.5. Paper outline

According to Euler's theorem, the general displacement of a rigid body with a fixed point is a rotation about some axis (Goldstein, 1980). Therefore, finite rotations can be represented by vectors having the axis direction and a magnitude describing the rotated angle. There are several possibilities ensuring a 1:1 correspondence between vectors and rotation orthogonal tensors, which hold in a finite domain. In the present paper, the rotation vector $\boldsymbol{\theta}$ will be adopted and, hence, all rotation related quantities will be expressed as a function of either $\boldsymbol{\theta}$ or its differential $\mathrm{d}\boldsymbol{\theta}$.

After introducing basic finite rotation concepts, the paper investigates rotation-dependent moments *in the finite rotation range*. Two main issues are dealt with:

(i) When is a rotation-dependent moment conservative?
(ii) In a conservative system with rotational degrees of freedom, when is the tangent operator symmetric?

Concerning the first question, after presenting a general theory based on the work-conjugacy between rotation-dependent moments and rotation measures, we establish (i) a *necessary condition* for a rotation-dependent moment to be conservative and (ii) the general form for conservative *isotropic* moment-rotation laws. Next, several particular rotation-dependent moment and rotation definitions are investigated, compared and characterized.

To answer the second question, we need to distinguish between *additive* and *multiplicative* updates of the rotation description. While the former always lead to symmetric tangent operators, the latter require the satisfaction of an additional condition, which is termed here as the *semi-tangential property*.

2. Basics of finite rotations

2.1. Rodrigues formula and the rotation vector

Finite rotations can be represented by an orthogonal tensor $\boldsymbol{\Lambda}$, an element of the rotation group

$$\mathrm{SO}(3) = \{\boldsymbol{\Lambda} : \mathbb{R}^3 \to \mathbb{R}^3 \text{ linear} \,|\, \boldsymbol{\Lambda}\boldsymbol{\Lambda}^{\mathrm{t}} = \mathbf{1} \wedge \det(\boldsymbol{\Lambda}) = 1\}$$

This tensor can be parameterized by the *rotation vector* $\boldsymbol{\theta}$, which is a vector aligned with the rotation axis and having a magnitude equal to the rotated angle θ. Rodrigues formula establishes the relation between $\boldsymbol{\theta}$ and $\boldsymbol{\Lambda}$ (e.g., see Goldstein, 1980, p. 164, or Crisfield, 1997, p. 191)

$$\boldsymbol{\Lambda} = \mathbf{1} + \frac{\sin\theta}{\theta}\tilde{\boldsymbol{\theta}} + \frac{1-\cos\theta}{\theta^2}\tilde{\boldsymbol{\theta}}^2 = \cos\theta\mathbf{1} + \frac{\sin\theta}{\theta}\tilde{\boldsymbol{\theta}} + \frac{1-\cos\theta}{\theta^2}\boldsymbol{\theta}\otimes\boldsymbol{\theta} \tag{1}$$

where $\tilde{\boldsymbol{\theta}}$ denotes the skew-symmetric second-order tensor for which $\boldsymbol{\theta}$ is the axial vector, meaning that

$$\tilde{\boldsymbol{\theta}}\boldsymbol{v} = \boldsymbol{\theta}\times\boldsymbol{v} \quad \forall\boldsymbol{v}\in\mathbb{R}^3 \tag{2}$$

We state here the following standard vector identities, which will be needed later:

$$\tilde{\boldsymbol{y}}\tilde{\boldsymbol{x}} = \boldsymbol{x}\otimes\boldsymbol{y} - (\boldsymbol{x}.\boldsymbol{y})\mathbf{1} \tag{3}$$

$$\widetilde{(\tilde{\boldsymbol{x}}\boldsymbol{y})} = \tilde{\boldsymbol{x}}\tilde{\boldsymbol{y}} - \tilde{\boldsymbol{y}}\tilde{\boldsymbol{x}} \tag{4}$$

2.2. Infinitesimal rotations and the spin

An expression for an infinitesimal rotation is obtained by considering only the linear terms in (1), yielding

$$\boldsymbol{\Lambda}_\epsilon = \mathbf{1} + \widetilde{\mathrm{d}\boldsymbol{\omega}} \tag{5}$$

where $\mathrm{d}\boldsymbol{\omega}$ is the infinitesimal rotation vector known as the spatial spin. If an infinitesimal rotation $\boldsymbol{\Lambda}_\epsilon$ is performed after a finite rotation $\boldsymbol{\Lambda}$, the resulting rotation tensor is obtained by the multiplicative update

$$\boldsymbol{\Lambda} + \mathrm{d}\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_\epsilon\boldsymbol{\Lambda} = \boldsymbol{\Lambda} + \widetilde{\mathrm{d}\boldsymbol{\omega}}\boldsymbol{\Lambda} \tag{6}$$

This means that

$$\mathrm{d}\boldsymbol{\Lambda} = \widetilde{\mathrm{d}\boldsymbol{\omega}}\boldsymbol{\Lambda} \tag{7}$$

It can be shown (Ritto-Corrêa and Camotim, 2002) that the axial vector $\mathrm{d}\boldsymbol{\omega}$ of the skew-symmetric tensor $\widetilde{\mathrm{d}\boldsymbol{\omega}}$ is given by

$$\mathrm{d}\boldsymbol{\omega} = \boldsymbol{T}\,\mathrm{d}\boldsymbol{\theta} \tag{8}$$

where

$$\boldsymbol{T} = \mathbf{1} + \frac{1-\cos\theta}{\theta^2}\tilde{\boldsymbol{\theta}} + \frac{\theta-\sin\theta}{\theta^3}\tilde{\boldsymbol{\theta}}^2 \tag{9}$$

The inverse of $\boldsymbol{T}$ reads (see, e.g., Ibrahimbegović et al., 1995)

$$\boldsymbol{T}^{-1} = \mathbf{1} - \frac{1}{2}\tilde{\boldsymbol{\theta}} + \frac{1}{\theta^2}\left(1 - \frac{(1+\cos\theta)\theta}{2\sin\theta}\right)\tilde{\boldsymbol{\theta}}^2 \tag{10}$$

an expression that breaks down for $\theta = 2\pi$.

2.3. The work performed by a moment

Upon an infinitesimal rotation $\mathrm{d}\boldsymbol{\omega}$, a moment $\boldsymbol{M}$ performs the infinitesimal work

$$\mathrm{d}W = \boldsymbol{M}\cdot\mathrm{d}\boldsymbol{\omega} \tag{11}$$

It is well known that the spin $\mathrm{d}\boldsymbol{\omega}$ is not a total differential, which means that there exists no '$\boldsymbol{\omega}$' from which $\mathrm{d}\boldsymbol{\omega}$ can be derived. Defining the *axial moment* $\boldsymbol{M}_\omega$ as a moment with fixed spatial components, it becomes obvious that the infinitesimal work $\mathrm{d}W = \boldsymbol{M}_\omega\cdot\mathrm{d}\boldsymbol{\omega}$ is not a total differential either.

3. General theory of rotation-dependent moments

3.1. Work-conjugacy

A rotation-dependent moment can be written in the general form

$$\boldsymbol{M} = \boldsymbol{Q}_a \boldsymbol{M}_a \tag{12}$$

where $\boldsymbol{M}_a$ is a constant vector defining the *initial* moment (the moment that acts when no rotation is present), $\boldsymbol{M}$ is the current value of the moment and $\boldsymbol{Q}_a = \boldsymbol{Q}_a(\boldsymbol{\Lambda})$ is a rotation-dependent second-order tensor that establishes the *moment-rotation law*, i.e. describes how the moment depends on the rotation.

We next consider a more general form of (11) and say that a moment $\boldsymbol{M}_a$ is *work-conjugate* to an infinitesimal rotation $\mathrm{d}\boldsymbol{a}$ if the work performed by the moment is

$$\mathrm{d}W = \boldsymbol{M}_a \cdot \mathrm{d}\boldsymbol{a} \tag{13}$$

where the infinitesimal rotation $\mathrm{d}\boldsymbol{a}$ may or may not be a total differential.

It is important to clarify that, at this stage, we do not specify neither how the vector $\mathrm{d}\boldsymbol{a}$ describes an infinitesimal rotation, nor the meaning of the moment $\boldsymbol{M}_a$. By proceeding in this way, we aim at establishing general expressions, which will be particularized later.

Let us assume that $\mathrm{d}\boldsymbol{a}$ is a total differential and that vector $\boldsymbol{M}_a$ is constant. Then, there exists a moment potential $V(\boldsymbol{a})$ such that

$$\mathrm{d}W = -\mathrm{d}V(\boldsymbol{a}) \quad \text{with } V(\boldsymbol{a}) = -\boldsymbol{M}_a \cdot \boldsymbol{a} \tag{14}$$

where $\boldsymbol{a}$ is a vector defining the rotation (in an yet unprescribed way). Therefore, $\boldsymbol{M}_a$ can be derived from $V(\boldsymbol{a})$,

$$\boldsymbol{M}_a = -\frac{\mathrm{d}V(\boldsymbol{a})}{\mathrm{d}\boldsymbol{a}} \tag{15}$$

and the work performed by moment $\boldsymbol{M} = \boldsymbol{Q}_a \boldsymbol{M}_a$, when the rotation measure $\boldsymbol{a}$ varies between $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, is

$$W = \int_1^2 \mathrm{d}W = -(V(\boldsymbol{a}_2) - V(\boldsymbol{a}_1)) \tag{16}$$

which implies that the moment generated by $\boldsymbol{M}_a$ and the by moment-rotation law $\boldsymbol{Q}_a$ is conservative. Using an abbreviated language to which we will resort often, we may say, alternatively, 'moment $\boldsymbol{M}_a$ is conservative'.

3.2. The rotation measures $\boldsymbol{a}$ and $\mathrm{d}\boldsymbol{a}$

Let us consider $\boldsymbol{a}$ to be related to the rotation vector $\boldsymbol{\theta}$ by

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{\theta}) \tag{17}$$

where we assume that $\boldsymbol{a}(\boldsymbol{\theta})$ establishes a 1:1 correspondence between $\boldsymbol{\theta}$ and $\boldsymbol{a}$, holding for a certain domain of $\boldsymbol{\theta}$. Differentiating (17), we obtain

$$\mathrm{d}\boldsymbol{a} = \boldsymbol{S}_a \mathrm{d}\boldsymbol{\theta} \tag{18}$$

where $\boldsymbol{S}_a$ is a second tensor given by

$$\boldsymbol{S}_a = \frac{\mathrm{d}\boldsymbol{a}(\boldsymbol{\theta})}{\mathrm{d}\boldsymbol{\theta}} \tag{19}$$

We remark that, although it may be impossible to write an expression like (17) for a given rotation definition, expression (18) is quite general. In particular, the d$\boldsymbol{a}$ appearing in Eq. (18) is *not* required to be a total differential.

Taking into account Eqs. (11)–(13), we can write

$$\mathrm{d}W = \boldsymbol{M} \cdot \mathrm{d}\boldsymbol{\omega} = \boldsymbol{Q}_a \boldsymbol{M}_a \cdot \mathrm{d}\boldsymbol{\omega} = \boldsymbol{M}_a \cdot \boldsymbol{Q}_a^{\mathrm{t}} \mathrm{d}\boldsymbol{\omega} = \boldsymbol{M}_a \cdot \mathrm{d}\boldsymbol{a} \tag{20}$$

which shows that

$$\mathrm{d}\boldsymbol{a} = \boldsymbol{Q}_a^{\mathrm{t}} \mathrm{d}\boldsymbol{\omega} \tag{21}$$

Observe that the meaning of $\boldsymbol{a}$ or d$\boldsymbol{a}$ can be apprehended from (17) or (21), respectively, for vector-like or spin-like rotation descriptions.

Furthermore, from Eqs. (18), (21) and (8), we obtain the relationship

$$\boldsymbol{S}_a = \boldsymbol{Q}_a^{\mathrm{t}} \boldsymbol{T} \tag{22}$$

3.3. A necessary condition for conservativeness

Assume that $\boldsymbol{M}_a$ is a conservative moment derived from a potential $V(\boldsymbol{a})$, as discussed in Section 3.1. Because we also assume a 1:1 correspondence between $\boldsymbol{a}$ and $\boldsymbol{\theta}$, this potential can be rewritten as

$$V(\boldsymbol{a}) = V(\boldsymbol{a}(\boldsymbol{\theta})) = V_a(\boldsymbol{\theta}) \tag{23}$$

Since the order in which two successive directional derivatives are performed is arbitrary, we have

$$D^2 V_a(\boldsymbol{\theta})[\mathrm{d}\boldsymbol{\theta}_1, \mathrm{d}\boldsymbol{\theta}_2] = D^2 V_a(\boldsymbol{\theta})[\mathrm{d}\boldsymbol{\theta}_2, \mathrm{d}\boldsymbol{\theta}_1] = \mathrm{d}\boldsymbol{\theta}_2 \cdot \boldsymbol{K}_a \mathrm{d}\boldsymbol{\theta}_1 \tag{24}$$

where $\boldsymbol{K}_a$ must be a *symmetric* second-order tensor.

To explicitly determine $\boldsymbol{K}_a$, we first use Eqs. (14) and (18) to yield

$$DV_a(\boldsymbol{\theta})[\mathrm{d}\boldsymbol{\theta}_1] = -\boldsymbol{M}_a \cdot \boldsymbol{S}_a \mathrm{d}\boldsymbol{\theta}_1 \tag{25}$$

Next, it is convenient to define an operator $\Xi_{DU}(\boldsymbol{u})$, associated with the directional derivative of a second-order tensor $\boldsymbol{U}$, through the relation

$$D\boldsymbol{U}[\mathrm{d}\boldsymbol{\theta}]\boldsymbol{u} = \Xi_{DU}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{\theta} \tag{26}$$

Taking into account that $\boldsymbol{M}_a$ is a constant vector, the second directional derivative of $V_a(\boldsymbol{\theta})$ is then given by

$$\begin{aligned} D^2 V_a(\boldsymbol{\theta})[\mathrm{d}\boldsymbol{\theta}_1, \mathrm{d}\boldsymbol{\theta}_2] &= -\boldsymbol{M}_a \cdot (D\boldsymbol{S}_a[\mathrm{d}\boldsymbol{\theta}_2])\,\mathrm{d}\boldsymbol{\theta}_1 = -(D\boldsymbol{S}_a^{\mathrm{t}}[\mathrm{d}\boldsymbol{\theta}_2])\boldsymbol{M}_a \cdot \mathrm{d}\boldsymbol{\theta}_1 = -\Xi_{D\boldsymbol{S}_a^{\mathrm{t}}}(\boldsymbol{M}_a)\,\mathrm{d}\boldsymbol{\theta}_2 \cdot \mathrm{d}\boldsymbol{\theta}_1 \\ &= -\mathrm{d}\boldsymbol{\theta}_2 \cdot \Xi^{\mathrm{t}}_{D\boldsymbol{S}_a^{\mathrm{t}}}(\boldsymbol{M}_a)\,\mathrm{d}\boldsymbol{\theta}_1 \end{aligned} \tag{27}$$

Finally, comparing (24) with (27), one finds that

$$\boldsymbol{K}_a = -\Xi^{\mathrm{t}}_{D\boldsymbol{S}_a^{\mathrm{t}}}(\boldsymbol{M}_a) \tag{28}$$

Hence, it follows that a *necessary condition* for a moment $\boldsymbol{M}_a$ to be *conservative* is the *symmetry* of $\Xi^{\mathrm{t}}_{D\boldsymbol{S}_a^{\mathrm{t}}}(\boldsymbol{M}_a)$. See Christoffersen (1989) and Saleeb et al. (1992) for other approaches to establish equivalent necessary conditions.

3.4. A class of problems—isotropic moment-rotation laws

Assume now that the moment-rotation law $\boldsymbol{Q}_a$ is an isotropic (tensor) function of $\boldsymbol{\Lambda}$, which means that the invariance requirement

$$\boldsymbol{R}\boldsymbol{Q}_a(\boldsymbol{\Lambda})\boldsymbol{R}^{\mathrm{t}} = \boldsymbol{Q}_a(\boldsymbol{R}\boldsymbol{\Lambda}\boldsymbol{R}^{\mathrm{t}}) \tag{29}$$

holds, where $\boldsymbol{R}$ is an arbitrary proper orthogonal tensor. Since $\boldsymbol{\Lambda}$ is itself an isotropic function of $\tilde{\boldsymbol{\theta}}$, a fact that follows easily from Eq. (1), then $\boldsymbol{Q}_a$ must be an isotropic function of the skew-symmetric tensor $\tilde{\boldsymbol{\theta}}$. Representations for the symmetric and skew-symmetric parts of $\boldsymbol{Q}_a$ can be found in Wang (1970), leading to [1]

$$\boldsymbol{Q}_a = q_0(\theta)\mathbf{1} + q_1(\theta)\tilde{\boldsymbol{\theta}} + q_2(\theta)\tilde{\boldsymbol{\theta}}^2 = \bar{q}_0(\theta)\mathbf{1} + q_1(\theta)\tilde{\boldsymbol{\theta}} + q_2(\theta)\boldsymbol{\theta}\otimes\boldsymbol{\theta} \tag{30}$$

with $\bar{q}_0(\theta) = q_0(\theta) - \theta^2 q_2(\theta)$.

Introducing (30) into Eq. (22), we find that

$$\boldsymbol{S}_a = s_0(\theta)\mathbf{1} + s_1(\theta)\tilde{\boldsymbol{\theta}} + s_2(\theta)\tilde{\boldsymbol{\theta}}^2 = \bar{s}_0(\theta)\mathbf{1} + s_1(\theta)\tilde{\boldsymbol{\theta}} + s_2(\theta)\boldsymbol{\theta}\otimes\boldsymbol{\theta} \tag{31}$$

where $\bar{s}_0(\theta) = s_0(\theta) - \theta^2 s_2(\theta)$ and

$$\begin{aligned}
s_0(\theta) &= q_0(\theta)\\
s_1(\theta) &= \frac{1-\cos\theta}{\theta^2}q_0(\theta) - \frac{\sin\theta}{\theta}q_1(\theta) - (1-\cos\theta)q_2(\theta)\\
s_2(\theta) &= \frac{\theta-\sin\theta}{\theta^3}q_0(\theta) - \frac{1-\cos\theta}{\theta^2}q_1(\theta) + \frac{\sin\theta}{\theta}q_2(\theta)
\end{aligned} \tag{32}$$

On the other hand, $\boldsymbol{Q}_a$ can be obtained from $\boldsymbol{S}_a$ using the inverse relationships

$$\begin{aligned}
q_0(\theta) &= s_0(\theta)\\
q_1(\theta) &= \frac{1}{2}s_0(\theta) - \frac{(1+\cos\theta)\theta}{2\sin\theta}s_1(\theta) - \frac{\theta^2}{2}s_2(\theta)\\
q_2(\theta) &= \frac{1}{\theta^2}\left(1 - \frac{(1+\cos\theta)\theta}{2\sin\theta}\right)s_0(\theta) - \frac{1}{2}s_1(\theta) + \frac{(1+\cos\theta)\theta}{2\sin\theta}s_2(\theta)
\end{aligned} \tag{33}$$

Observing that the directional derivative of θ is given by

$$\theta^2 = \boldsymbol{\theta}\cdot\boldsymbol{\theta} \;\Rightarrow\; 2\theta\,\mathrm{d}\theta = \boldsymbol{\theta}\cdot\mathrm{d}\boldsymbol{\theta} + \mathrm{d}\boldsymbol{\theta}\cdot\boldsymbol{\theta} \;\Rightarrow\; d\theta = \frac{\boldsymbol{\theta}\cdot\mathrm{d}\boldsymbol{\theta}}{\theta} \tag{34}$$

the transpose of the directional derivative of Eq. (31) is

$$D\boldsymbol{S}_a^{\mathrm{t}}[\mathrm{d}\boldsymbol{\theta}] = -s_1(\theta)\mathrm{d}\tilde{\boldsymbol{\theta}} + s_2(\theta)(\mathrm{d}\boldsymbol{\theta}\otimes\boldsymbol{\theta} + \boldsymbol{\theta}\otimes\mathrm{d}\boldsymbol{\theta}) + (\boldsymbol{\theta}\cdot\mathrm{d}\boldsymbol{\theta})\left(\frac{\bar{s}_0'(\theta)}{\theta}\mathbf{1} - \frac{s_1'(\theta)}{\theta}\tilde{\boldsymbol{\theta}} + \frac{s_2'(\theta)}{\theta}\boldsymbol{\theta}\otimes\boldsymbol{\theta}\right) \tag{35}$$

and its associate tensor reads (see also Ritto-Corrêa and Camotim, 2002)

$$\begin{aligned}
\boldsymbol{\Xi}_{D\boldsymbol{S}_a^{\mathrm{t}}}(\boldsymbol{M}_a) = {} & s_1(\theta)\tilde{\boldsymbol{M}}_a + s_2(\theta)(\boldsymbol{\theta}\cdot\boldsymbol{M}_a)\mathbf{1} + s_2(\theta)\boldsymbol{\theta}\otimes\boldsymbol{M}_a + \frac{\bar{s}_0'(\theta)}{\theta}\boldsymbol{M}_a\otimes\boldsymbol{\theta} - \frac{s_1'(\theta)}{\theta}(\tilde{\boldsymbol{\theta}}\boldsymbol{M}_a\otimes\boldsymbol{\theta})\\
& + \frac{s_2'(\theta)}{\theta}(\boldsymbol{\theta}\cdot\boldsymbol{M}_a)\boldsymbol{\theta}\otimes\boldsymbol{\theta}
\end{aligned} \tag{36}$$

Therefore, we have

$$\begin{aligned}
\boldsymbol{\Xi}^{\mathrm{t}}_{D\boldsymbol{S}_a^{\mathrm{t}}}(\boldsymbol{M}_a) = {} & s_2(\theta)(\boldsymbol{\theta}\cdot\boldsymbol{M}_a)\mathbf{1} + s_2(\theta)\boldsymbol{M}_a\otimes\boldsymbol{\theta} + \frac{\bar{s}_0'(\theta)}{\theta}\boldsymbol{\theta}\otimes\boldsymbol{M}_a + \frac{s_2'(\theta)}{\theta}(\boldsymbol{\theta}\cdot\boldsymbol{M}_a)\boldsymbol{\theta}\otimes\boldsymbol{\theta} - s_1(\theta)\tilde{\boldsymbol{M}}_a\\
& - \frac{s_1'(\theta)}{\theta}(\boldsymbol{\theta}\otimes\tilde{\boldsymbol{\theta}}\boldsymbol{M}_a)
\end{aligned} \tag{37}$$

[1] We acknowledge the contribution of Prof. H. Xiao, who kindly helped us to properly derive the $\boldsymbol{Q}_a$ representation.

which, for arbitrary $\boldsymbol{M_a}$ and $\boldsymbol{\theta}$ vectors, is a symmetric expression only if the two following conditions hold:

$$s_2(\theta) = \frac{\bar{s}_0'(\theta)}{\theta} \quad s_1(\theta) = 0 \tag{38}$$

From the result obtained in Section 3.3, we infer that, if $\boldsymbol{Q_a}$ is such that conditions (38) *do not* hold, the associated moment $\boldsymbol{M_a}$ is *not* conservative.

On the other hand, if conditions (38) do hold, it is easy to find a relationship between $\boldsymbol{a}$ and $\boldsymbol{\theta}$, which has the form

$$\boldsymbol{a}(\boldsymbol{\theta}) = \bar{s}_0(\theta)\boldsymbol{\theta} \tag{39}$$

In fact, the differentiation of this expression yields

$$\mathrm{d}\boldsymbol{a} = \bar{s}_0(\theta)\,\mathrm{d}\boldsymbol{\theta} + \frac{\bar{s}_0'(\theta)}{\theta}(\boldsymbol{\theta}\cdot\mathrm{d}\boldsymbol{\theta})\boldsymbol{\theta} = \left(\bar{s}_0(\theta)\mathbf{1} + \frac{\bar{s}_0'(\theta)}{\theta}\boldsymbol{\theta}\otimes\boldsymbol{\theta}\right)\mathrm{d}\boldsymbol{\theta} \tag{40}$$

which, in face of (18), shows that

$$\boldsymbol{S_a} = \bar{s}_0(\theta)\mathbf{1} + \frac{\bar{s}_0'(\theta)}{\theta}\boldsymbol{\theta}\otimes\boldsymbol{\theta} \tag{41}$$

This expression has the form of (31) and satisfies the conditions (38).

Hence, for an isotropic moment-rotation law, (38) are both necessary and sufficient conditions for $\boldsymbol{M_a}$ to be conservative. Moreover, the infinitesimal rotation d$\boldsymbol{a}$ is a total differential and the explicit relation between $\boldsymbol{a}$ and $\boldsymbol{\theta}$ is of the form (39).

Finally, introducing the conservativeness conditions (38) into Eqs. (33), we obtain the general expression for an isotropic moment-rotation law

$$\boldsymbol{Q_a} = (\bar{s}_0(\theta) + \theta\bar{s}_0'(\theta))\mathbf{1} + \frac{\bar{s}_0(\theta)}{2}\tilde{\boldsymbol{\theta}} + \frac{1}{\theta^2}\left(\bar{s}_0(\theta) - \frac{(1+\cos\theta)\theta}{2\sin\theta}\bar{s}_0(\theta) + \theta\bar{s}_0'(\theta)\right)\tilde{\boldsymbol{\theta}}^2 \tag{42}$$

where only the exact definition of the scalar function $\bar{s}_0(\theta)$ is left open. This expression can also be written as

$$\boldsymbol{Q_a} = \frac{(1+\cos\theta)\theta}{2\sin\theta}\bar{s}_0(\theta)\mathbf{1} + \frac{\bar{s}_0(\theta)}{2}\tilde{\boldsymbol{\theta}} + \frac{1}{\theta^2}\left(\bar{s}_0(\theta) - \frac{(1+\cos\theta)\theta}{2\sin\theta}\bar{s}_0(\theta) + \theta\bar{s}_0'(\theta)\right)\boldsymbol{\theta}\otimes\boldsymbol{\theta} \tag{43}$$

3.5. First-order expressions and the semi-tangential property

Argyris et al. (1979a) were primarily concerned with the linear terms of the moment-rotation law, as can be understood from the following quote:

> Our task is to investigate the changes occurring in the moments when the rigid levers used for the generation of the moments are subject to rotations. It suffices in doing so to consider only small rotations since the resulting linear terms of the changes determine the second-order terms of the total potential energy of the finite elements required for the derivation of the tangent stiffness. (Argyris et al., 1979a, p. 36)

Let us assume $\boldsymbol{M_a}$ to be a rotation-dependent moment which is both conservative and isotropic. Observe first that, since $\boldsymbol{M} = \boldsymbol{M_a}$ for $\boldsymbol{\theta} = \mathbf{0}$, $q_0(0) = \bar{q}_0(0) = 1$, which implies that (see Eq. (32))

$$s_0(0) = \bar{s}_0(0) = 1 \tag{44}$$

Secondly, observe that the conservative conditions (38) imply that, for $\boldsymbol{\theta} = \mathbf{0}$, one has

$$\bar{s}_0'(0) = 0 \quad s_1(0) = 0 \tag{45}$$

Hence, from (31) and (42) one finds that, for a conservative moment-rotation law, the first-order approximations of $\boldsymbol{S}_{\boldsymbol{a}}$ and $\boldsymbol{Q}_{\boldsymbol{a}}$ are given by

$$\boldsymbol{S}_{\boldsymbol{a}} = \mathbf{1} + \mathrm{O}(\boldsymbol{\theta}^2) \tag{46}$$

$$\boldsymbol{Q}_{\boldsymbol{a}} = \mathbf{1} + \tfrac{1}{2}\tilde{\boldsymbol{\theta}} + \mathrm{O}(\boldsymbol{\theta}^2) \tag{47}$$

Since the $\boldsymbol{Q}_{\boldsymbol{a}}$ expression coincides with the first-order approximations of Ziegler and Argyris semi-tangential moment definitions, we will characterize all the moment rotation-laws sharing the first-order approximation (47) as having the *semi-tangential property*. Similarly, all the infinitesimal rotational measures $\mathrm{d}\boldsymbol{a}$ sharing the first-order approximation (46) will also be designated as *semi-tangential*.

4. A study of particular cases

In this section, we address and discuss several possible kinds of moments and rotations. In particular, the general formulae developed in the previous section are used to establish work-conjugate pairs. One deals mainly with isotropic moment-rotation laws, so as to be able to use the results obtained in Section 3.4, but the anisotropic Ziegler's pseudo-tangential moment is also considered. On the other hand, Ziegler's semi-tangential and quasi-tangential moments (Ziegler, 1968) are not dealt with, due to the difficulties in establishing their moment-rotation law in the finite rotation range, as already mentioned in the introduction.

4.1. The rotation vector and the rotational moment

Let us start with the simplest case, in which we have $\boldsymbol{a} = \boldsymbol{\theta}$, i.e. $\boldsymbol{a}$ is the *rotation vector*. Then it trivially follows that

$$\boldsymbol{S}_{\boldsymbol{\theta}} = \mathbf{1} \quad \boldsymbol{Q}_{\boldsymbol{\theta}} = \boldsymbol{T}^{-\mathrm{t}} \tag{48}$$

Thus, an applied moment $\boldsymbol{M} = \boldsymbol{T}^{-\mathrm{t}}\boldsymbol{M}_{\boldsymbol{\theta}}$, where $\boldsymbol{M}_{\boldsymbol{\theta}}$ is a constant vector, is conservative. In Ritto-Corrêa and Camotim (2002), we have designated $\boldsymbol{M}_{\boldsymbol{\theta}}$ as the *rotational moment*.

4.2. The spatial spin and the axial moment

The simplest spin-like rotation description is the *spatial spin*, in which the individual rotation axes are fixed in space and the composition of two successive rotations is given by $\boldsymbol{\Lambda}_{2\circ 1} = \boldsymbol{\Lambda}_2\boldsymbol{\Lambda}_1$. In this case, $\mathrm{d}\boldsymbol{a} = \mathrm{d}\boldsymbol{\omega}$ and we have

$$\boldsymbol{S}_{\boldsymbol{\omega}} = \boldsymbol{T} \quad \boldsymbol{Q}_{\boldsymbol{\omega}} = \boldsymbol{T}^{-\mathrm{t}}\boldsymbol{T}^{\mathrm{t}} = \mathbf{1} \tag{49}$$

which just confirms the known fact that the (non-conservative) *axial moment* $\boldsymbol{M}_{\boldsymbol{\omega}}$ is rotation independent.

4.3. The material spin and the follower moment

We now consider rotations about follower axes, i.e., axes attached to the body (Argyris, 1982), for which the composition of two successive rotations is given by $\boldsymbol{\Lambda}_{2\circ 1} = \boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_2$. This means that, when performing a new rotation, its rotation vector must be first rotated through all the previous rotations, i.e.,

$$\mathrm{d}\boldsymbol{\omega} = \boldsymbol{\Lambda}\,\mathrm{d}\boldsymbol{\Omega} \tag{50}$$

The spin vector $\mathrm{d}\boldsymbol{\Omega}$ is generally known as the *material spin*. Comparing Eqs. (21) and (50), and using the fact that $\boldsymbol{\Lambda}^{-\mathrm{t}} = \boldsymbol{\Lambda}$, we find that

$$\boldsymbol{Q}_{\boldsymbol{\Omega}} = \boldsymbol{\Lambda} \tag{51}$$

Hence, $\boldsymbol{M} = \boldsymbol{\Lambda}\boldsymbol{M}_{\boldsymbol{\Omega}}$ and we conclude that the moments work-conjugate to the material spins are *follower* (*or tangential*) *moments*.

Finally, from Eq. (22) and using the equality $\boldsymbol{T}^{\mathrm{t}} = \boldsymbol{\Lambda}^{\mathrm{t}}\boldsymbol{T}$ (see, e.g., Ibrahimbegović et al., 1995; Ritto-Corrêa and Camotim, 2002), we obtain

$$\boldsymbol{S}_{\boldsymbol{\Omega}} = \boldsymbol{\Lambda}^{\mathrm{t}}\boldsymbol{T} = \boldsymbol{T}^{\mathrm{t}} \tag{52}$$

This expression does not pass the conservativeness test defined by (38) and, therefore, follower moments $\boldsymbol{M}_{\boldsymbol{\Omega}}$ are not conservative.

4.4. The cross semi-tangential moment

The cross semi-tangential moment stems from Argyris construction (Argyris et al., 1979a, p. 37), which is shown in Fig. 2. The initial moment is generated via two orthogonal unit levers, having the directions of the two unit vectors $\boldsymbol{l}$ and $\boldsymbol{k}$, and four forces equal to $(M_0/2)\boldsymbol{l}$, $-(M_0/2)\boldsymbol{l}$, $(M_0/2)\boldsymbol{k}$ and $-(M_0/2)\boldsymbol{k}$. Thus, the initial moment is

$$\boldsymbol{M}_c = \boldsymbol{l} \times \frac{M_0\boldsymbol{k}}{2} - \boldsymbol{k} \times \frac{M_0\boldsymbol{l}}{2} = M_0\boldsymbol{m} \tag{53}$$

with

$$\boldsymbol{m} = \boldsymbol{l} \times \boldsymbol{k} \tag{54}$$

Rotating now the levers through $\boldsymbol{\theta}$, while keeping the forces constant, we obtain the new moment as

$$\boldsymbol{M} = (\boldsymbol{\Lambda}\boldsymbol{l}) \times \frac{M_0\boldsymbol{k}}{2} - (\boldsymbol{\Lambda}\boldsymbol{k}) \times \frac{M_0\boldsymbol{l}}{2} \tag{55}$$

This expression can be rewritten as

$$\boldsymbol{M} = \frac{M_0}{2}(-\tilde{\boldsymbol{k}}\boldsymbol{\Lambda}\boldsymbol{l} + \tilde{\boldsymbol{l}}\boldsymbol{\Lambda}\boldsymbol{k}) \tag{56}$$

or, after introducing Eq. (1) and rearranging terms, as

$$\boldsymbol{M} = \frac{M_0}{2}(\tilde{\boldsymbol{l}}\boldsymbol{k} - \tilde{\boldsymbol{k}}\boldsymbol{l}) - \frac{\sin\theta}{2\theta}M_0(\tilde{\boldsymbol{l}}\tilde{\boldsymbol{k}} - \tilde{\boldsymbol{k}}\tilde{\boldsymbol{l}})\boldsymbol{\theta} + \frac{1-\cos\theta}{2\theta^2}M_0(\tilde{\boldsymbol{l}}\tilde{\boldsymbol{\theta}}^2\boldsymbol{k} - \tilde{\boldsymbol{k}}\tilde{\boldsymbol{\theta}}^2\boldsymbol{l}) \tag{57}$$

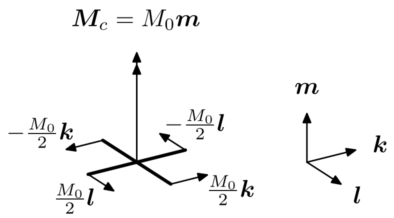

Fig. 2. Cross semi-tangential moment and unit vector orientation.

From Eqs. (54) and (4), one finds that

$$\boldsymbol{m} = \frac{\tilde{\boldsymbol{l}}\boldsymbol{k} - \tilde{\boldsymbol{k}}\boldsymbol{l}}{2} \tag{58}$$

$$\tilde{\boldsymbol{m}} = \tilde{\boldsymbol{l}}\tilde{\boldsymbol{k}} - \tilde{\boldsymbol{k}}\tilde{\boldsymbol{l}} \tag{59}$$

which allows us to write the expressions inside the first two parentheses of (57) solely in terms of $\boldsymbol{m}$. As for the expression inside the last parentheses, it can be transformed, through the use of Eq. (3), into

$$\begin{aligned}\tilde{\boldsymbol{l}}\tilde{\boldsymbol{\theta}}^2\boldsymbol{k} - \tilde{\boldsymbol{k}}\tilde{\boldsymbol{\theta}}^2\boldsymbol{l} &= \tilde{\boldsymbol{l}}(\boldsymbol{\theta}\otimes\boldsymbol{\theta})\boldsymbol{k} - \theta^2\tilde{\boldsymbol{l}}\boldsymbol{k} - \tilde{\boldsymbol{k}}(\boldsymbol{\theta}\otimes\boldsymbol{\theta})\boldsymbol{l} + \theta^2\tilde{\boldsymbol{k}}\boldsymbol{l} = -\tilde{\boldsymbol{\theta}}(\boldsymbol{l}\otimes\boldsymbol{k})\boldsymbol{\theta} + \tilde{\boldsymbol{\theta}}(\boldsymbol{k}\otimes\boldsymbol{l})\boldsymbol{\theta} - \theta^2(\tilde{\boldsymbol{l}}\boldsymbol{k} - \tilde{\boldsymbol{k}}\boldsymbol{l}) \\ &= \tilde{\boldsymbol{\theta}}(-\tilde{\boldsymbol{k}}\tilde{\boldsymbol{l}} + (\boldsymbol{k}\cdot\boldsymbol{l})\mathbf{1} + \tilde{\boldsymbol{l}}\tilde{\boldsymbol{k}} - (\boldsymbol{k}\cdot\boldsymbol{l})\mathbf{1})\boldsymbol{\theta} - 2\theta^2\boldsymbol{m} = \tilde{\boldsymbol{\theta}}\tilde{\boldsymbol{m}}\boldsymbol{\theta} - 2\theta^2\boldsymbol{m} = -\tilde{\boldsymbol{\theta}}^2\boldsymbol{m} - 2\theta^2\boldsymbol{m}\end{aligned} \tag{60}$$

Then, Eq. (57) becomes

$$\boldsymbol{M} = M_0\boldsymbol{m} - \frac{\sin\theta}{2\theta}M_0\tilde{\boldsymbol{m}}\boldsymbol{\theta} - \frac{1-\cos\theta}{2\theta^2}M_0\tilde{\boldsymbol{\theta}}^2\boldsymbol{m} - \frac{1-\cos\theta}{\theta^2}M_0\theta^2\boldsymbol{m} \tag{61}$$

which, in face of Eq. (53), yields the desired moment rotation law for the cross semi-tangential moment,

$$\boldsymbol{M} = \boldsymbol{Q}_c\boldsymbol{M}_c \tag{62}$$

$$\boldsymbol{Q}_c = \cos\theta\mathbf{1} + \frac{\sin\theta}{2\theta}\tilde{\boldsymbol{\theta}} - \frac{1-\cos\theta}{2\theta^2}\tilde{\boldsymbol{\theta}}^2 \tag{63}$$

Matrix $\boldsymbol{Q}_c$ was previously obtained by Saleeb et al. (1992), although it was expressed in the alternative form $\boldsymbol{Q}_c = \frac{1}{2}(\mathrm{tr}(\boldsymbol{\Lambda})\mathbf{1} - \boldsymbol{\Lambda}^{\mathrm{t}})$ and no derivations details were provided.

The expression for $\boldsymbol{S}_c$ is readily obtained, with the help of Eqs. (32), as

$$\boldsymbol{S}_c = \frac{\sin\theta}{\theta}\mathbf{1} + \frac{\theta\cos\theta - \sin\theta}{\theta^3}\boldsymbol{\theta}\otimes\boldsymbol{\theta} \tag{64}$$

Now, since it turns out that

$$\frac{\left(\frac{\sin\theta}{\theta}\right)'}{\theta} = \frac{\theta\cos\theta - \sin\theta}{\theta^3} \tag{65}$$

holds, (64) has precisely the form of (41). Hence, we confirm that the cross semi-tangential moment is conservative and, furthermore, we conclude that

$$\boldsymbol{c} = \frac{\sin\theta}{\theta}\boldsymbol{\theta} \tag{66}$$

which, interestingly, is a rotational parameterization considered previously by Pietraszkiewicz and Badur (1983). In other words, *the sine-scaled rotation vector is the rotation measure work-conjugate to the cross semi-tangential moment* (which is the only semi-tangential moment considered by Argyris). As far as we know, this is an original result.

4.5. The pseudo-rotation and the pseudo-rotational moment

We have just seen how a conservative moment-rotation law can lead to a rotational measure of the form (17). It is also possible to perform the reverse operation, i.e. to start from a rotational measure and obtain the corresponding conservative moment-rotation law. In order to achieve this, we only need to define the function $\bar{s}_0(\theta)$ and to use expressions (38) and (33). To illustrate this procedure, consider the rotational measure

$$\boldsymbol{q} = \frac{\tan\frac{\theta}{2}}{\frac{\theta}{2}}\boldsymbol{\theta} = \frac{2\sin\theta}{(1+\cos\theta)\theta}\boldsymbol{\theta} \tag{67}$$

which was termed 'parameter pseudo-vector' in (Argyris, 1982). For this reason, the rotation described by $\boldsymbol{q}$ will be called here *pseudo-rotation* and its corresponding moment *pseudo-rotational moment*.

From Eq. (67) and the conservativeness conditions (38), we find that

$$\boldsymbol{S}_q = \frac{2\sin\theta}{(1+\cos\theta)\theta}\mathbf{1} + \frac{2(\theta-\sin\theta)}{(1+\cos\theta)\theta^3}\boldsymbol{\theta}\otimes\boldsymbol{\theta} \tag{68}$$

which, with the help of Eqs. (33), reveals that

$$\boldsymbol{Q}_q = \frac{2}{1+\cos\theta}\mathbf{1} + \frac{\sin\theta}{(1+\cos\theta)\theta}\tilde{\boldsymbol{\theta}} + \frac{(1-\cos\theta)}{(1+\cos\theta)\theta^2}\tilde{\boldsymbol{\theta}}^2 \tag{69}$$

4.6. The mean semi-tangential moment

Argyris et al. (1979a) did not carry out their work up to the derivation of (63). Having dealt with first-order approximations of the moment-rotation law, they found instead that

> ... we may understand the semi-tangential moment as a mean value between an axial and a follower moment. (Argyris et al., 1979a, p. 38)

While this statement is correct in the context of a first-order approximation, for our present purpose—to investigate the full finite rotation range—it is preferable to assign a different name to this moment. We then choose the designation *mean semi-tangential moment* to represent the moment-rotation law

$$\boldsymbol{M} = \frac{\mathbf{1}+\boldsymbol{\Lambda}}{2}\boldsymbol{M}_m \tag{70}$$

This means that

$$\boldsymbol{Q}_m = \frac{\mathbf{1}+\boldsymbol{\Lambda}}{2} = \mathbf{1} + \frac{\sin\theta}{2\theta}\tilde{\boldsymbol{\theta}} + \frac{1-\cos\theta}{2\theta^2}\tilde{\boldsymbol{\theta}}^2 \tag{71}$$

which only matches (63) up to the linear terms. From Eqs. (71) and (32), we also find that

$$\boldsymbol{S}_m = \frac{\sin\theta}{\theta}\mathbf{1} + \frac{\theta-\sin\theta}{\theta^3}\boldsymbol{\theta}\otimes\boldsymbol{\theta} \tag{72}$$

an expression which does not pass the conservativeness test (38). Thus, the infinitesimal *mean semi-tangential rotation* $\mathrm{d}\boldsymbol{m} = \boldsymbol{S}_m\mathrm{d}\boldsymbol{\theta}$ is not a total differential and the mean moment $\boldsymbol{M}_m$ is not conservative.

4.7. Argyris commutative semi-tangential rotations

Argyris et al. (Argyris et al., 1978, 1979a; Argyris, 1982) state that (their) semi-tangential moments are work-conjugated to the *semi-tangential rotations*, defined in

> After a preceding semi-tangential rotation v_p the 'rotation vector' v_p of a subsequent semi-tangential rotation is changed to the vectorial mean of the initial 'vector' v_s and the 'vector' v_{sR} which results from a rotation of v_s through v_p. (Argyris et al., 1978, p. 428)

These rotations will be here designated as *commutative semi-tangential rotations*, because their composition is commutative (for two arbitrary rotations or for three rotations about orthogonal axes, see Argyris et al., 1978; Argyris, 1982; Kim et al., 2001). Such definition leads to [2]

$$\mathrm{d}\boldsymbol{\omega} = \frac{\mathbf{1} + \boldsymbol{\Lambda}}{2}\,\mathrm{d}\boldsymbol{s} \tag{73}$$

where $\mathrm{d}\boldsymbol{s}$ is the axial vector of an infinitesimal commutative semi-tangential rotation. To find the conjugate pair of $\mathrm{d}\boldsymbol{s}$, we compare Eqs. (73) and (21) to conclude that

$$\boldsymbol{Q}_s^{-\mathrm{t}} = \frac{\mathbf{1} + \boldsymbol{\Lambda}}{2} \tag{74}$$

Inverting and transposing this equation yields

$$\boldsymbol{Q}_s = \mathbf{1} + \frac{\sin\theta}{(1+\cos\theta)\theta}\tilde{\boldsymbol{\theta}} \tag{75}$$

and, after applying (32), one finds that

$$\boldsymbol{S}_s = \frac{2\sin\theta}{(1+\cos\theta)\theta}\mathbf{1} + \frac{1}{\theta^2}\left(1 - \frac{2\sin\theta}{(1+\cos\theta)\theta}\right)\boldsymbol{\theta}\otimes\boldsymbol{\theta} \tag{76}$$

Since this expression fails to pass the conservativeness test (38), the *commutative semi-tangential moment* $\boldsymbol{M}_s$ (defined as the work-conjugate of the commutative semi-tangential rotation) is not conservative. However, it is interesting to note the similarities between Eqs. (68) and (76), related to the (conservative) pseudo-rotational moment and the (non-conservative) commutative semi-tangential moment.

4.8. The half semi-tangential moment

Argyris definition of a semi-tangential rotation is not always consistently the same, as can be seen from the following quote:

> . . . in any rotation of a system, the axis of a subsequent semi-tangential rotation is rotated through one half of the angle through which the system itself is rotated. (Argyris et al., 1979a, p. 42)

We will name the rotation associated to this definition as the *half semi-tangential rotation*. We then have

$$\mathrm{d}\boldsymbol{\omega} = \sqrt{\boldsymbol{\Lambda}}\,\mathrm{d}\boldsymbol{h} \tag{77}$$

where $\sqrt{\boldsymbol{\Lambda}}$ is the orthogonal tensor satisfying

$$\boldsymbol{\Lambda} = \sqrt{\boldsymbol{\Lambda}}\sqrt{\boldsymbol{\Lambda}} \tag{78}$$

and given by

$$\sqrt{\boldsymbol{\Lambda}} = \mathbf{1} + \frac{\sin\frac{\theta}{2}}{\theta}\tilde{\boldsymbol{\theta}} + \frac{1-\cos\frac{\theta}{2}}{\theta^2}\tilde{\boldsymbol{\theta}}^2 \tag{79}$$

The corresponding composition rule is $\boldsymbol{\Lambda}_{2\circ 1} = \sqrt{\boldsymbol{\Lambda}_1}\boldsymbol{\Lambda}_2\sqrt{\boldsymbol{\Lambda}_1}$. In this case, from Eqs. (21) and (77) and the fact that $\sqrt{\boldsymbol{\Lambda}}$ is an orthogonal matrix, we conclude that

[2] The ‘rotation vector’ referred to in Argyris et al. (1978) was scaled by $\tan(\theta/2)/\theta$, a distinction which is not important because Eq. (73) involves an (additional) *infinitesimal* rotation.

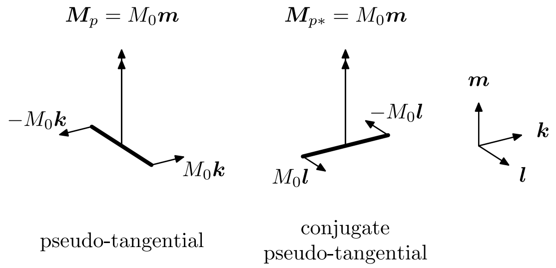

Fig. 3. Pseudo semi-tangential moment and its conjugate.

$$\boldsymbol{Q_h} = \sqrt{\boldsymbol{\Lambda}} \tag{80}$$

By using (32), one then finds

$$\boldsymbol{S_h} = \frac{2\sin\frac{\theta}{2}}{\theta}\mathbf{1} + \frac{1}{\theta^2}\left(1 - \frac{2\sin\frac{\theta}{2}}{\theta}\right)\boldsymbol{\theta}\otimes\boldsymbol{\theta} \tag{81}$$

which, by failing to pass the conservativeness test (38), shows that the *half semi-tangential moment* is not conservative.

4.9. Ziegler's pseudo-tangential moment

Finally, let us address one anisotropic moment-rotation law, namely Ziegler's conservative pseudo-tangential moment, [3] shown in Fig. 3. The initial moment is

$$\boldsymbol{M_p} = M_0\boldsymbol{l}\times\boldsymbol{k} = M_0\boldsymbol{m} \tag{82}$$

while, after the lever is rotated through $\boldsymbol{\theta}$, the new moment is (the identity $\boldsymbol{l} = \boldsymbol{k}\times\boldsymbol{m}$ is used)

$$\boldsymbol{M} = M_0(\boldsymbol{\Lambda l})\times\boldsymbol{k} = -M_0\tilde{\boldsymbol{k}}\boldsymbol{\Lambda l} = -M_0\tilde{\boldsymbol{k}}\boldsymbol{\Lambda}\tilde{\boldsymbol{k}}\boldsymbol{m} = -\tilde{\boldsymbol{k}}\boldsymbol{\Lambda}\tilde{\boldsymbol{k}}\boldsymbol{M_p} \tag{83}$$

This means that the moment-rotation law is

$$\boldsymbol{Q_p} = -\tilde{\boldsymbol{k}}\boldsymbol{\Lambda}\tilde{\boldsymbol{k}} \tag{84}$$

Since the results derived in Section 3.4 are not applicable to the anisotropic $\boldsymbol{Q_p}$, the work-conjugate rotation measure is best obtained by writing directly the work expression. The initial and rotated positions of the two forces, with respect to the center of the lever, are $\pm\boldsymbol{l}/2$ and $\pm\boldsymbol{\Lambda l}/2$. Then, the work performed by the two forces is

$$W = 2\frac{\boldsymbol{\Lambda l}-\boldsymbol{l}}{2}\cdot M_0\boldsymbol{k} = (\boldsymbol{\Lambda}-\mathbf{1})\boldsymbol{l}\cdot M_0\boldsymbol{k} = M_0\left(\frac{\sin\theta}{\theta}\tilde{\boldsymbol{\theta}}\boldsymbol{l}\cdot\boldsymbol{k} + \frac{1-\cos\theta}{\theta^2}\tilde{\boldsymbol{\theta}}^2\boldsymbol{l}\cdot\boldsymbol{k}\right) \tag{85}$$

Performing the following manipulations:

$$\tilde{\boldsymbol{\theta}}\boldsymbol{l}\cdot\boldsymbol{k} = -\tilde{\boldsymbol{l}}\boldsymbol{\theta}\cdot\boldsymbol{k} = \boldsymbol{\theta}\cdot\tilde{\boldsymbol{l}}\boldsymbol{k} = \boldsymbol{\theta}\cdot\boldsymbol{m} \tag{86}$$

[3] Note that several authors designate Ziegler's pseudo-tangential moment as 'quasi-tangential' (Argyris et al., 1978, 1979a; Yang and Kuo, 1994).

$$\tilde{\boldsymbol{\theta}}^2\boldsymbol{l}\cdot\boldsymbol{k} = \tilde{\boldsymbol{\theta}}^2\tilde{\boldsymbol{k}}\boldsymbol{m}\cdot\boldsymbol{k} = -\boldsymbol{m}\cdot\tilde{\boldsymbol{k}}\tilde{\boldsymbol{\theta}}^2\boldsymbol{k} = -\tilde{\boldsymbol{k}}(\boldsymbol{\theta}\otimes\boldsymbol{\theta}-\theta^2\mathbf{1})\boldsymbol{k}\cdot\boldsymbol{m} = -(\boldsymbol{\theta}\cdot\boldsymbol{k})\tilde{\boldsymbol{k}}\boldsymbol{\theta}\cdot\boldsymbol{m} + \theta^2\tilde{\boldsymbol{k}}\boldsymbol{k}\cdot\boldsymbol{m}$$
$$= -(\boldsymbol{\theta}\cdot\boldsymbol{k})\tilde{\boldsymbol{k}}\boldsymbol{\theta}\cdot\boldsymbol{m} \quad (87)$$

the work can be expressed as

$$W = \boldsymbol{M}_p\cdot\boldsymbol{p} \quad (88)$$

in which $\boldsymbol{p}$ is a rotational measure given by [4]

$$\boldsymbol{p} = \frac{\sin\theta}{\theta}\boldsymbol{\theta} - \frac{1-\cos\theta}{\theta^2}(\boldsymbol{\theta}\cdot\boldsymbol{k})\tilde{\boldsymbol{k}}\boldsymbol{\theta} \quad (89)$$

It is interesting to consider also the 'conjugate' pseudo-tangential moment $\boldsymbol{M}_{p^*}$ (see Fig. 3), which, after a similar derivation is carried out, can be characterized by

$$\boldsymbol{Q}_{p^*} = -\tilde{\boldsymbol{l}}\boldsymbol{\Lambda}\tilde{\boldsymbol{l}} \quad (90)$$

$$\boldsymbol{p}^* = \frac{\sin\theta}{\theta}\boldsymbol{\theta} + \frac{1-\cos\theta}{\theta^2}(\boldsymbol{\theta}\cdot\boldsymbol{k})\tilde{\boldsymbol{k}}\boldsymbol{\theta} \quad (91)$$

At last, observe that, although $\boldsymbol{p}$ and $\boldsymbol{p}^*$ are both anisotropic functions of $\boldsymbol{\theta}$, their average $\boldsymbol{c} = (\boldsymbol{p}+\boldsymbol{p}^*)/2$ is isotropic. The same can be said of moment $\boldsymbol{M}_c = (\boldsymbol{M}_p + \boldsymbol{M}_{p^*})/2$, as shown in Section 4.4.

4.10. Synthesis

Table 1 displays the $\boldsymbol{S}_a$ and $\boldsymbol{Q}_a$ tensors corresponding to all the addressed isotropic work-conjugate finite rotation measures and rotation-dependent moment pairs.

Argyris and co-workers (Argyris et al., 1978, 1979a) have dealt with the cross semi-tangential, mean semi-tangential, commutative semi-tangential and half semi-tangential concepts (under the common designation 'semi-tangential') as if they were all equivalent. The explanation for this fact is that, as already mentioned at the beginning of Section 3.5, these concepts were used in the context of small (incremental semi-tangential) rotations. Indeed, by looking at the series expansions of the operators $\boldsymbol{Q}_a$ shown in Table 2, it becomes clear that all the above moment definitions are equivalent up to the first order but differ in the higher-order terms. Furthermore, the pseudo-rotational and rotational moments also have the same first-order approximation, thus sharing the *semi-tangential* property.

Fig. 4 depicts, for each investigated isotropic moment-rotation law, how an initial horizontal moment changes when the body rotates anti-clockwise (2π) about an axis perpendicular to the plane of the page. Each successive moment vector represents the effect of an additional $\pi/6$ rotation. One observes that:

(i) For this particular rotation—perpendicular to the initial moment—the cross and mean semi-tangential moments behave exactly in the same way. The same can be said about the pseudo-rotational and commutative semi-tangential moments. However, no such coincidence would occur for a rotation parallel to the initial moment vector.
(ii) The rotational moment becomes infinite for a full 2π rotation. On the other hand, the pseudo-rotational and commutative semi-tangential moments become infinite for $\theta = \pi$.
(iii) All the semi-tangential moments (all but the axial and follower moments) display very similar initial behaviors, but differ significantly afterwards.

[4] Note that the expression for $\boldsymbol{p}$ is not uniquely determined, since, in face of (88), the addition of a term perpendicular to $\boldsymbol{m}$ does not affect the value of W. A similar remark can be made about the moment-rotation law $\boldsymbol{Q}_p$.

Table 1
Kinds of isotropic rotations and rotation-dependent moments

Kind	$\boldsymbol{a}$ or d$\boldsymbol{a}$	$\boldsymbol{S}_{\boldsymbol{a}}$	$\boldsymbol{Q}_{\boldsymbol{a}}$
Axial	$\mathrm{d}\boldsymbol{\omega}$	$\frac{\sin\theta}{\theta}\mathbf{1}+\frac{1-\cos\theta}{\theta^2}\tilde{\boldsymbol{\theta}}+\frac{\theta-\sin\theta}{\theta^3}\boldsymbol{\theta}\otimes\boldsymbol{\theta}$	$\mathbf{1}$
Rotational	$\boldsymbol{\theta}$	$\mathbf{1}$	$\mathbf{1}+\frac{1}{2}\tilde{\boldsymbol{\theta}}+\frac{1}{\theta^2}\left(1-\frac{(1+\cos\theta)\theta}{2\sin\theta}\right)\tilde{\boldsymbol{\theta}}^2$
Cross semi-tangential	$\boldsymbol{c}=\frac{\sin\theta}{\theta}\boldsymbol{\theta}$	$\frac{\sin\theta}{\theta}\mathbf{1}+\frac{\theta\cos\theta-\sin\theta}{\theta^3}\boldsymbol{\theta}\otimes\boldsymbol{\theta}$	$\cos\theta\mathbf{1}+\frac{\sin\theta}{2\theta}\tilde{\boldsymbol{\theta}}-\frac{1-\cos\theta}{2\theta^2}\tilde{\boldsymbol{\theta}}^2$
Mean semi-tangential	$\mathrm{d}\boldsymbol{m}$	$\frac{\sin\theta}{\theta}\mathbf{1}+\frac{\theta-\sin\theta}{\theta^3}\boldsymbol{\theta}\otimes\boldsymbol{\theta}$	$\mathbf{1}+\frac{\sin\theta}{2\theta}\tilde{\boldsymbol{\theta}}+\frac{1-\cos\theta}{2\theta^2}\tilde{\boldsymbol{\theta}}^2$
Pseudo-rotational	$\boldsymbol{q}=\frac{2\sin\theta}{(1+\cos\theta)\theta}\boldsymbol{\theta}$	$\frac{2\sin\theta}{(1+\cos\theta)\theta}\mathbf{1}+\frac{2(\theta-\sin\theta)}{(1+\cos\theta)\theta^3}\boldsymbol{\theta}\otimes\boldsymbol{\theta}$	$\frac{2}{1+\cos\theta}\mathbf{1}+\frac{\sin\theta}{(1+\cos\theta)\theta}\tilde{\boldsymbol{\theta}}+\frac{(1-\cos\theta)}{(1+\cos\theta)\theta^2}\tilde{\boldsymbol{\theta}}^2$
Commutative semi-tangential	$\mathrm{d}\boldsymbol{s}$	$\frac{2\sin\theta}{(1+\cos\theta)\theta}\mathbf{1}+\frac{1}{\theta^2}\left(1-\frac{2\sin\theta}{(1+\cos\theta)\theta}\right)\boldsymbol{\theta}\otimes\boldsymbol{\theta}$	$\mathbf{1}+\frac{\sin\theta}{(1+\cos\theta)\theta}\tilde{\boldsymbol{\theta}}$
Half semi-tangential	$\mathrm{d}\boldsymbol{h}$	$\frac{2\sin\frac{\theta}{2}}{\theta}\mathbf{1}+\frac{1}{\theta^2}\left(1-\frac{2\sin\frac{\theta}{2}}{\theta}\right)\boldsymbol{\theta}\otimes\boldsymbol{\theta}$	$\mathbf{1}+\frac{\sin\frac{\theta}{2}}{\theta}\tilde{\boldsymbol{\theta}}+\frac{1-\cos\frac{\theta}{2}}{\theta^2}\tilde{\boldsymbol{\theta}}^2$
Follower	$\mathrm{d}\boldsymbol{\Omega}$	$\frac{\sin\theta}{\theta}\mathbf{1}-\frac{1-\cos\theta}{\theta^2}\tilde{\boldsymbol{\theta}}+\frac{\theta-\sin\theta}{\theta^3}\boldsymbol{\theta}\otimes\boldsymbol{\theta}$	$\mathbf{1}+\frac{\sin\theta}{\theta}\tilde{\boldsymbol{\theta}}+\frac{1-\cos\theta}{\theta^2}\tilde{\boldsymbol{\theta}}^2$

Table 2
Expansion of $\boldsymbol{Q}_{\boldsymbol{a}}$

Moment	Term of order				
	0	1	2	3	4
Axial	$\mathbf{1}$	–	–	–	–
Rotational	$\mathbf{1}$	$\frac{1}{2}\tilde{\boldsymbol{\theta}}$	$\frac{1}{12}\tilde{\boldsymbol{\theta}}^2$	–	$\frac{\theta^2}{720}\tilde{\boldsymbol{\theta}}^2$
Cross semi-tangential	$\mathbf{1}$	$\frac{1}{2}\tilde{\boldsymbol{\theta}}$	$-\frac{\theta^2}{2}\mathbf{1}-\frac{1}{4}\tilde{\boldsymbol{\theta}}^2$	$-\frac{\theta^2}{12}\tilde{\boldsymbol{\theta}}$	$\frac{\theta^4}{24}\mathbf{1}+\frac{\theta^2}{48}\tilde{\boldsymbol{\theta}}^2$
Mean semi-tangential	$\mathbf{1}$	$\frac{1}{2}\tilde{\boldsymbol{\theta}}$	$\frac{1}{4}\tilde{\boldsymbol{\theta}}^2$	$-\frac{\theta^2}{12}\tilde{\boldsymbol{\theta}}$	$-\frac{\theta^2}{48}\tilde{\boldsymbol{\theta}}^2$
Pseudo-rotational	$\mathbf{1}$	$\frac{1}{2}\tilde{\boldsymbol{\theta}}$	$\frac{\theta^2}{4}\mathbf{1}+\frac{1}{4}\tilde{\boldsymbol{\theta}}^2$	$\frac{\theta^2}{24}\tilde{\boldsymbol{\theta}}$	$\frac{\theta^4}{24}\mathbf{1}+\frac{\theta^2}{24}\tilde{\boldsymbol{\theta}}^2$
Commutative semi-tangential	$\mathbf{1}$	$\frac{1}{2}\tilde{\boldsymbol{\theta}}$	–	$\frac{\theta^2}{24}\tilde{\boldsymbol{\theta}}$	–
Half semi-tangential	$\mathbf{1}$	$\frac{1}{2}\tilde{\boldsymbol{\theta}}$	$\frac{1}{8}\tilde{\boldsymbol{\theta}}^2$	$-\frac{\theta^2}{48}\tilde{\boldsymbol{\theta}}$	$-\frac{\theta^2}{384}\tilde{\boldsymbol{\theta}}^2$
Follower	$\mathbf{1}$	$\tilde{\boldsymbol{\theta}}$	$\frac{1}{2}\tilde{\boldsymbol{\theta}}^2$	$-\frac{\theta^2}{6}\tilde{\boldsymbol{\theta}}$	$-\frac{\theta^2}{24}\tilde{\boldsymbol{\theta}}^2$

Finally, in Table 3 the addressed moment-rotation laws are classified according to the following properties:

Finite—when there is a known moment-rotation law valid in the finite rotation range.
Conservative—when the moment $\boldsymbol{M}_{\boldsymbol{a}}$ is conservative.
Isotropic—when the moment-rotation law $\boldsymbol{Q}_{\boldsymbol{a}}$ is isotropic.
Isometric—when the moment $\boldsymbol{M}=\boldsymbol{Q}_{\boldsymbol{a}}\boldsymbol{M}_{\boldsymbol{a}}$ has constant magnitude. In this case, $\boldsymbol{Q}_{\boldsymbol{a}}$ must be an orthogonal tensor.
Mechanistic—when it is possible to identify a (simple) mechanism originating the moment. Obviously, this property is somewhat fuzzy, as it relies on the capacity to devise such a mechanism.
Semi-tangential—when the moment-rotation law $\boldsymbol{Q}_{\boldsymbol{a}}$ shares the semi-tangential property, i.e. has the first-order approximation given by Eq. (47).
Cyclic—when, after a full circle rotation ($\theta=2\pi$), the moment-rotation law yields a moment equal to the initial one. In this case, the moment-rotation law should feature only the unit vector $\boldsymbol{\theta}/\theta$ and trigonometric functions of θ.

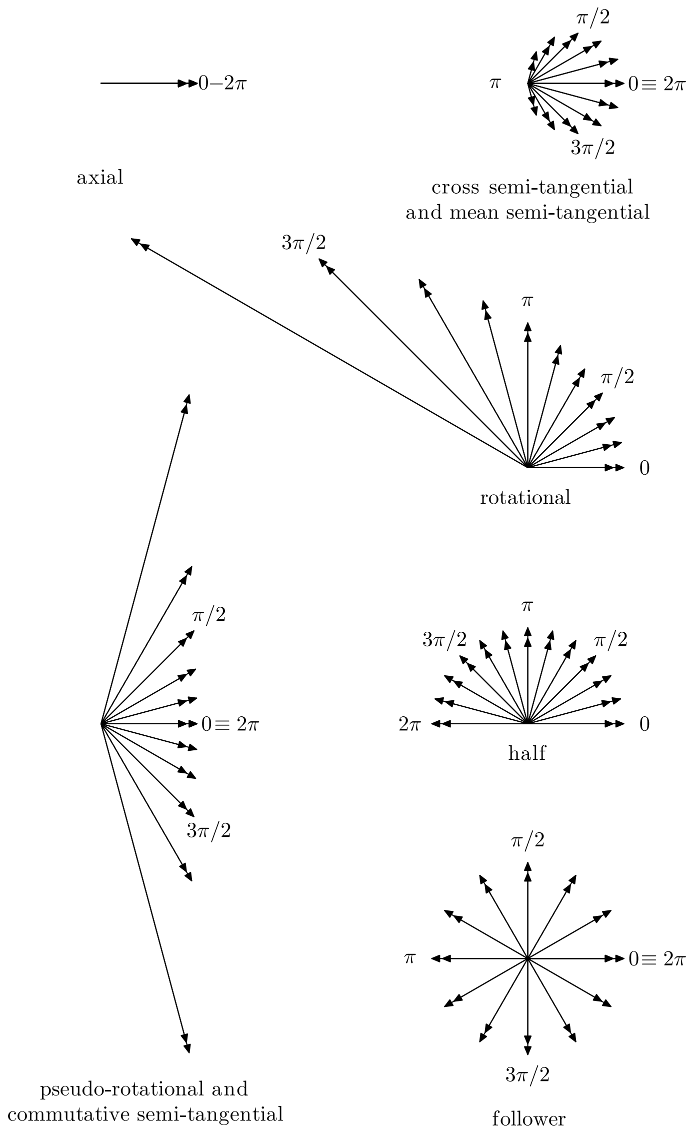

Fig. 4. Isotropic rotation-dependent moments. The initial moment is horizontal (to the right). Each successive moment corresponds to an additional $\pi/6$ anti-clockwise rotation perpendicular to the plane of the page.

Parallel-invariant—when moment $\boldsymbol{M}$ does not change for rotations $\boldsymbol{\theta}$ about an axis parallel to the initial moment $\boldsymbol{M}_a$. In this case, term $\mathbf{1}$ in $\boldsymbol{Q}_a$ must have an unit coefficient.

Singularity value—the smallest θ value for which the moment-rotation law yields an infinite $\boldsymbol{M}$ magnitude.

Table 3 also includes Ziegler's semi-tangential and quasi-tangential moment definitions (Ziegler, 1968), which were not investigated in detail here. Since there are no known moment-rotation laws for Ziegler's quasi- and semi-tangential moments in the finite rotation range, some fields are left unanswered.

Table 3
Properties of several kinds of rotation-dependent moments

Kind	Finite	Conservative	Isotropic	Isometric	Mechanistic	Semi-tangential	Cyclic	Parallel-invariant	Singularity value
Axial	✓	–	✓	✓	✓	–	✓	✓	–
Ziegler semi-tangential	–	✓	✓	–	✓	✓	?	✓	?
Ziegler quasi-tangential	–	✓	–	–	✓	–	?	✓	?
Ziegler pseudo-tangential	✓	✓	–	–	✓	–	✓	✓	–
Rotational	✓	✓	✓	–	–	✓	–	✓	2π
Cross semi-tangential	✓	✓	✓	–	✓	✓	✓	–	–
Mean semi-tangential	✓	–	✓	–	–	✓	✓	✓	–
Pseudo-rotational	✓	✓	✓	–	–	✓	✓	–	π
Commutative semi-tangential	✓	–	✓	–	–	✓	✓	✓	π
Half semi-tangential	✓	–	✓	✓	–	✓	–	✓	–
Follower	✓	–	✓	✓	✓	–	✓	✓	–

5. On the symmetry of the tangent operator

5.1. The tangent operator

We now turn our attention to the second issue raised in the paper outline: In a conservative system with rotational degrees of freedom, when is the tangent operator symmetric?

Since the system is assumed to be conservative, let us denote its potential energy by Π. To keep the presentation simple, we address the case in which only one 3D rotation is involved. This is sufficient because the potential non-symmetry of the tangent operator (in a conservative system) stems from the interaction between the three components within each 3D rotation, and not from the one between different 3D rotations.

Using the principle of stationary potential energy, equilibrium can then be stated as

$$\delta\Pi = \delta\boldsymbol{a}\cdot\Pi_{,\boldsymbol{a}} = \mathbf{0} \tag{92}$$

The linearization of $\delta\Pi$, at the current position, can be written as

$$\delta\Pi = \delta\Pi_{\mathbf{0}} + \Delta\delta\Pi_{\mathbf{0}} + \cdots \tag{93}$$

in which the zero subscript identifies the quantities evaluated at the current position. The linear term can be expressed as

$$\Delta\delta\Pi_{\mathbf{0}} = \delta\boldsymbol{a}\cdot[\Pi_{,\boldsymbol{aa}}]_{\mathbf{0}}\,\Delta\boldsymbol{a} \tag{94}$$

and our task is to discuss the conditions required for the tangent operator $[\Pi_{,\boldsymbol{aa}}]_{\mathbf{0}}$ to be symmetric.

First of all, it is necessary to know how the rotational degrees of freedom describe the rotation. In particular, it is important (i) to select an infinitesimal rotation measure $\mathrm{d}\boldsymbol{a}$ and (ii) to choose either an *additive* or a *multiplicative* update of the orthogonal tensor. [5] Note that the same rotation measure can be used in both update schemes. For example, in the context of the formulation of geometrically exact beam finite elements, the rotation vector has been used in either additive (Cardona and Geradin, 1988; Ibrahimbegović et al., 1995; Ritto-Corrêa and Camotim, 2002) or multiplicative (Buechter and Ramm, 1992) updates.

[5] It is also possible to describe rotations with more than three parameters (e.g. the four Euler parameters), subjected to some constraints. Such *constrained updates* are not dealt with here.

Table 4
Tensor functions $\boldsymbol{\Lambda}(\boldsymbol{a})$

$\boldsymbol{\Lambda}(\boldsymbol{a})$	Range	1:1 Range
$\boldsymbol{\Lambda}(\boldsymbol{\theta}) = \mathbf{1} + \frac{\sin\theta}{\theta}\tilde{\boldsymbol{\theta}} + \frac{1-\cos\theta}{\theta^2}\tilde{\boldsymbol{\theta}}^2$	$\boldsymbol{\theta} \in]-\infty, +\infty[$	$\boldsymbol{\theta} \in]-\pi, +\pi]$
$\boldsymbol{\Lambda}(\boldsymbol{q}) = \mathbf{1} + \frac{1}{1+\frac{q^2}{4}}\left(\tilde{\boldsymbol{q}} + \frac{\tilde{\boldsymbol{q}}^2}{2}\right)$	$\boldsymbol{q} \in]-\infty, +\infty[$ $\boldsymbol{\theta} \in]-\pi, +\pi[$	$\boldsymbol{\theta} \in]-\pi, +\pi[$
$\boldsymbol{\Lambda}(\boldsymbol{c}) = \mathbf{1} + \tilde{\boldsymbol{c}} + \frac{1-\sqrt{1-c^2}}{c^2}\tilde{\boldsymbol{c}}^2$	$\boldsymbol{c} \in [-1, +1]$ $\boldsymbol{\theta} \in \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$	$\boldsymbol{\theta} \in \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$

5.2. Additive updates

Consider a generic update, in which $\boldsymbol{\Lambda}_{\mathrm{O}}$ and $\boldsymbol{\Lambda}_{\mathrm{N}}$ designate the 'old' and 'new' orthogonal tensors. In an *additive update*, we define an orthogonal tensor function $\boldsymbol{\Lambda}(\boldsymbol{a})$, on the basis of which the old and new orthogonal tensors are given by

$$\boldsymbol{\Lambda}_{\mathrm{O}} = \boldsymbol{\Lambda}(\boldsymbol{a}_{\mathrm{O}}) \quad \boldsymbol{\Lambda}_{\mathrm{N}} = \boldsymbol{\Lambda}(\boldsymbol{a}_{\mathrm{N}}) = \boldsymbol{\Lambda}(\boldsymbol{a}_{\mathrm{O}} + \Delta\boldsymbol{a}) \tag{95}$$

Notice that an additive update obviously depends on the existence of (i) a total rotation measure $\boldsymbol{a}$ and (ii) a corresponding tensor function $\boldsymbol{\Lambda}(\boldsymbol{a})$. Table 4 displays the expressions of $\boldsymbol{\Lambda}(\boldsymbol{a})$, for the (isotropic and finite) rotation measures considered in Section 4. The first expression, $\boldsymbol{\Lambda}(\boldsymbol{\theta})$, is just Rodrigues formula, while the others can be obtained by expressing $\boldsymbol{\theta}$ in terms of $\boldsymbol{c}$ or $\boldsymbol{q}$ (see also Argyris, 1982; Pietraszkiewicz and Badur, 1983; Crisfield, 1997). It is well known that a parameterization of $\boldsymbol{\Lambda}$ involving only three parameters cannot be both global and singularity free (Stuelpnagel, 1964) and, for that reason, two ranges are also displayed in Table 4: (i) the range of application of the formula $\boldsymbol{\Lambda}(\boldsymbol{a})$ and (ii) the range in which such formula provides a 1:1 correspondence between $\boldsymbol{a}$ and $\boldsymbol{\Lambda}$. The rotation vector $\boldsymbol{\theta}$ appears to be the best choice for an additive update because (i) it has a simple geometric meaning, (ii) it has an unlimited range and a maximal 1:1 range (see Table 4) and (iii) its work-conjugate moment only becomes infinite for $\theta = 2\pi$ (see Table 3). Its only (minor) drawback is the presence of trigonometric functions which makes the linearization task difficult, but not impossible as shown by the authors (Ritto-Corrêa and Camotim, 2002).

The use of an additive update in a conservative system implies necessarily a symmetric tangent operator, since the mere existence of a rotation measure $\boldsymbol{a}$ ensures that the potential energy can be written as $\Pi(\boldsymbol{a})$ and, therefore,

$$\Pi_{,\boldsymbol{a}} = \sum_{i=1}^{3} \frac{\mathrm{d}\Pi(\boldsymbol{a})}{\mathrm{d}a_i}\boldsymbol{e}_i \quad \Pi_{,\boldsymbol{aa}} = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\mathrm{d}^2\Pi(\boldsymbol{a})}{\mathrm{d}a_i \mathrm{d}a_j}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{96}$$

where $\boldsymbol{e}_i$ stands for one of the three unit base vectors of a Cartesian coordinate system. From (96) it becomes obvious that the tangent operator $\Pi_{,\boldsymbol{aa}}$ must be symmetric for an additive update.

5.3. Multiplicative updates

In a *multiplicative update*, an orthogonal tensor function $\boldsymbol{\Lambda}(\Delta\boldsymbol{a})$ defines the transformation between the old and new orthogonal tensors through

$$\boldsymbol{\Lambda}_{\mathrm{N}} = \boldsymbol{\Lambda}(\Delta\boldsymbol{a})\boldsymbol{\Lambda}_{\mathrm{O}} \tag{97}$$

Here, the 'increment' $\Delta\boldsymbol{a}$ is just the finite counterpart of the infinitesimal $\mathrm{d}\boldsymbol{a}$ corresponding to the selected rotation description. After each finite update, $\boldsymbol{\Lambda}_{\mathrm{N}}$ becomes $\boldsymbol{\Lambda}_{\mathrm{O}}$ and $\Delta\boldsymbol{a}$ is reset to $\mathbf{0}$ which implies $\boldsymbol{\Lambda}(\Delta\boldsymbol{a}) = \mathbf{1}$ (a null rotation).

Since $\boldsymbol{\Lambda} = \mathbf{1}$ corresponds to $\boldsymbol{\theta} = \mathbf{0}$ and to $\boldsymbol{T} = \boldsymbol{Q}_a = \boldsymbol{S}_a = \mathbf{1}$, all d$\boldsymbol{a}$ coincide with the spatial spin d$\boldsymbol{\omega}$ and, thus, the multiplicative update (97) is basically independent of the meaning attributed to d$\boldsymbol{a}$. [6] It then might appear that, in a multiplicative update, the nature of d$\boldsymbol{a}$ is irrelevant. However, when evaluating the tangent operator, two linearizations are performed (one virtual and one incremental). Hence, even if $\boldsymbol{S}_a = \mathbf{1}$ at the beginning of each update, its derivative is not necessarily null, which is of paramount importance to the symmetry of the tangent operator, as shown next.

Since the system is deemed conservative, we can write the potential energy as $\Pi(\boldsymbol{\theta})$ where $\boldsymbol{\theta}$ denotes the rotation vector measured from the *current* position (this means that the current position corresponds to $\boldsymbol{\theta} = \mathbf{0}$). From Eq. (18), we may then express the variations of $\boldsymbol{\theta}$ in terms of $\delta\boldsymbol{a}$ and $\Delta\boldsymbol{a}$,

$$\delta\boldsymbol{\theta} = \boldsymbol{S}_a^{-1}\delta\boldsymbol{a} \quad \Delta\boldsymbol{\theta} = \boldsymbol{S}_a^{-1}\Delta\boldsymbol{a} \tag{98}$$

In addition, let us introduce the following quantities:

$$\Pi_{,\theta} = \sum_{i=1}^{3} \frac{\mathrm{d}\Pi(\boldsymbol{\theta})}{\mathrm{d}\theta_i}\boldsymbol{e}_i \quad \Pi_{,\theta\theta} = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\mathrm{d}^2\Pi(\boldsymbol{\theta})}{\mathrm{d}\theta_i\,\mathrm{d}\theta_j}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{99}$$

To obtain the expressions for $\Pi_{,a}$ and $\Pi_{,aa}$, we first write $\delta\Pi$ and $\Delta\delta\Pi$ in terms of $\delta\boldsymbol{\theta}$ and $\Delta\boldsymbol{\theta}$ and then make use of Eqs. (98),

$$\delta\Pi = \delta\boldsymbol{\theta}\cdot\Pi_{,\theta} = \boldsymbol{S}_a^{-1}\delta\boldsymbol{a}\cdot\Pi_{,\theta} = \delta\boldsymbol{a}\cdot\boldsymbol{S}_a^{-\mathrm{t}}\Pi_{,\theta} \tag{100}$$

$$\begin{aligned}\Delta\delta\Pi &= \delta\boldsymbol{a}\cdot D\boldsymbol{S}_a^{-\mathrm{t}}[\Delta\boldsymbol{\theta}]\Pi_{,\theta} + \delta\boldsymbol{a}\cdot\boldsymbol{S}_a^{-\mathrm{t}}\Pi_{,\theta\theta}\Delta\boldsymbol{\theta} = \delta\boldsymbol{a}\cdot\boldsymbol{\Xi}_{D\boldsymbol{S}_a^{-\mathrm{t}}}(\Pi_{,\theta})\Delta\boldsymbol{\theta} + \delta\boldsymbol{a}\cdot\boldsymbol{S}_a^{-\mathrm{t}}\Pi_{,\theta\theta}\Delta\boldsymbol{\theta}\\ &= \delta\boldsymbol{a}\cdot\boldsymbol{\Xi}_{D\boldsymbol{S}_a^{-\mathrm{t}}}(\Pi_{,\theta})\boldsymbol{S}_a^{-1}\Delta\boldsymbol{a} + \delta\boldsymbol{a}\cdot\boldsymbol{S}_a^{-\mathrm{t}}\Pi_{,\theta\theta}\boldsymbol{S}_a^{-1}\Delta\boldsymbol{a}\end{aligned} \tag{101}$$

which means that

$$\Pi_{,a} = \boldsymbol{S}_a^{-\mathrm{t}}\Pi_{,\theta} \tag{102}$$

$$\Pi_{,aa} = \boldsymbol{\Xi}_{D\boldsymbol{S}_a^{-\mathrm{t}}}(\Pi_{,\theta})\boldsymbol{S}_a^{-1} + \boldsymbol{S}_a^{-\mathrm{t}}\Pi_{,\theta\theta}\boldsymbol{S}_a^{-1} \tag{103}$$

The tangent operator, evaluated at the current position ($\boldsymbol{\theta} = \mathbf{0}$ and, therefore, also $\boldsymbol{S}_a = \mathbf{1}$), reads then

$$[\Pi_{,aa}]_{\mathbf{0}} = [\boldsymbol{\Xi}_{D\boldsymbol{S}_a^{-\mathrm{t}}}(\Pi_{,\theta}) + \Pi_{,\theta\theta}]_{\theta=\mathbf{0}} \tag{104}$$

While the second term is clearly symmetric, the first one depends on the specific choice of d$\boldsymbol{a}$, namely on the nature of $\boldsymbol{S}_a^{-\mathrm{t}}$. Observe, however, that the first term becomes null at an equilibrium point where one has $\Pi_{,\theta} = \mathbf{0}$ (see Eq. (92)). This fact is in accordance with a similar finding of Simo and Vu-Quoc (1986), obtained in the context of Reissner–Simo beam theory.

Assuming now that the chosen d$\boldsymbol{a}$ is semi-tangential, the corresponding $\boldsymbol{S}_a$ has no linear term in $\boldsymbol{\theta}$ (see Eq. (46)), thus implying that linear terms are also absent from $\boldsymbol{S}_a^{-\mathrm{t}}$. Hence, both $D\boldsymbol{S}_a^{-\mathrm{t}}[\Delta\boldsymbol{\theta}]$ and $\boldsymbol{\Xi}_{D\boldsymbol{S}_a^{-\mathrm{t}}}(\Pi_{,\theta})$ are null for $\boldsymbol{\theta} = \mathbf{0}$, which means that *semi-tangential multiplicative updates* lead (in conservative systems) to *symmetric* tangent operators. Furthermore, as the evaluation of $\Delta\delta\Pi$ takes place at $\boldsymbol{\theta} = \mathbf{0}$, the higher-order terms of $\boldsymbol{S}_a$ are irrelevant and, therefore, *all* semi-tangential multiplicative updates lead to the *same* tangent operator.

These findings are in accordance with the known fact that multiplicative updates based on either d$\boldsymbol{s}$ (Argyris et al., 1978, 1979a) or d$\boldsymbol{\theta}$ (Buechter and Ramm, 1992) lead to symmetric tangent operators, as both these rotation measures share the semi-tangential property.

[6] Nonetheless, there is some freedom in building the orthogonal tensor function $\boldsymbol{\Lambda}(\Delta\boldsymbol{a})$, as it is only required that its first-order approximation is $\boldsymbol{\Lambda}(\Delta\boldsymbol{a}) = \mathbf{1} + \widetilde{\Delta\boldsymbol{a}}$.

6. Conclusion

Ziegler (1968) and Argyris et al. (1978) have introduced the concept of rotation-dependent moments, but with distinct motivations. The former was concerned with conservative applied moments, the latter with the symmetry of the tangent operator in systems with rotational degrees of freedom and subjected to conservative loadings.

Obtaining the moment-rotation law for a given applied moment requires the knowledge of the moment generation mechanism and is a task that must be performed on a case-to-case basis (as illustrated here for the cross semi-tangential moment and Ziegler's pseudo-tangential moment).

For an applied moment to be conservative, its moment-rotation law must satisfy some conditions. A general necessary condition for conservativeness was identified. Furthermore, for the particular case of isotropic moment-rotation laws, a straightforward conservativeness test was devised. The application of such test has shown that the rotational, the pseudo-rotational and the cross semi-tangential moments are conservative, while all the other moments dealt with in this work are not. In addition, it was found that all isotropic conservative moments share the same first-order approximation, thus providing the grounds to establish a new (more general) definition for the semi-tangential property.

In what concerns the rotation description, it is important to distinguish between additive and multiplicative updates.

In an additive update, finite rotations must be described by vectors $\boldsymbol{a}$. For a conservative system, it is then possible to write the potential energy in terms of the variables $\boldsymbol{a}$, thereby ensuring that the tangent operator is symmetric. The rotation vector is arguably the best choice for an additive update because (i) it has a simple geometric meaning and (ii) its 'good-behavior' range is the widest among all the rotation measures.

In a multiplicative update, the variation of the orthogonal tensor is described by a spin-like variable $\mathrm{d}\boldsymbol{a}$ (not necessarily a total differential) and the tangent operator is, in general, unsymmetrical. However, the adoption of a semi-tangential multiplicative update always leads to a symmetric tangent operator.

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